BOUND STATES IN QUANTUM FIELD THEORY, SCALAR FIELDS

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The main aim of this paper is to demonstrate the method called "the Bosonization of Nonlocal Currents" (BNC), used for calculations of bound states in a quark model, within the simplest relativistic quantum field model of two scalar fields with the Yukawa type interaction:

$$L = \Phi^+(\Box - M^2)\Phi + \frac{1}{2}\phi(\Box - m^2)\phi + g\phi\Phi^+\Phi.$$

A second aim is to clarify the relation between BNC and two widely used methods, employed in recent particle physics to calculate bound states of interacting particles, based on the nonrelativistic Schrödinger equation (the S-method), and the relativistic Bethe-Salpeter equation (the BS-method), and to determine the conditions on parameters of a quantum field model dictating a definite method to be applied.

It is shown that all these methods can be applied only in the weak coupling regime (the effective dimensionless coupling constant λ should be less than 1). The basic parameter separating the relativistic and nonrelativistic pictures is $\xi = \frac{m}{M}$, namely, $\xi \ll 1$ with $\lambda \ll 1$ leads to the potential picture, i.e., the bound state is described by the nonrelativistic Schrödinger equation. For $\xi \geq 1$ and $\lambda \leq 1$ the Schrödinger potential picture is not valid and the Bethe-Salpeter equation or the BNC should be used, where the BNC method has a slightly bit wider region of applicability.

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I. INTRODUCTION.

Great efforts have been made to understand how bound states arise in the formalism of quantum field theory and to work out effective methods to calculate all characteristics of these bound states, especially their masses. Unfortunately, we observe that there is no well-defined unique method, like the Schrödinger equation in nonrelativistic quantum mechanics, which can be used practically for any problems of nonrelativistic quantum physics. We can conclude that QFT of today is not well suited to describe bound state problem, (see, for example, [1]).

The analysis of a bound state is simplest when the constituent particles can be considered to be nonrelativistic, i.e., when they travel at speeds considerably less than c. The physical evident criterion to tell that a bound state is nonrelativistic is for the binding energy to be small compared to the rest energies of the constituents. The theoretical criterion is that the coupling constants should to be weak and masses of intermediate particles (photon in QED, mesons in nuclear physics, gluons in QCD) should be small in comparison with masses of constituents.

The best example is hydrogen-like systems, which can be considered nonrelativistic, and the experiments with great accuracy were required to develop the methods to calculate next relativistic corrections (see, for example, [1–3]).

The situation in nuclear and particle physics is completely different. First, the coupling constants are not small any more. Second, in nuclear physics, although the binding energy is relatively small in comparison with nucleon masses, the masses of intermediate mesons realizing the strong nuclear interaction are not small. In particular, the most adequate description of the deuteron can be done by the Bethe-Salpeter equation where the contribution of all light mesons should be taken into account (see, for example, [4]).

In particle physics, only hadrons made out of heavy quarks can be considered by the nonrelativistic potential methods although the decays of heavy hadrons into light ones require relativistic methods to describe these transformations. The most familiar light-quark states are intrinsically relativistic, so that they require pure relativistic methods. Besides, they are constituted at distances where the confinement phenomenon should be taken into account. In addition, one may ask whether the free Dirac equation applicable to describe the light quarks in this region? Therefore, using the nonrelativistic Schrödinger equation to describe the light-quark systems by fitting parameters of potentials (see, for example, [5,6]) can be considered heroic attempt to understand light meson physics by unsuitable methods in a very rough theoretical approximation.

In relativistic quantum field theory, bound states are identified by the occurrence of poles of corresponding amplitudes or Green functions with appropriate quantum numbers. These poles have a nonperturbative character, so that they can arise as a result of a nonperturbative rearrangement of series over a coupling constant. The investigation of nonperturbative properties was done by establishing integral equations among amplitudes and Green functions, using

the specific structure of a Lagrangian. One should say that these equations, having absolutely general form, in reality can be used when the kernels contain contributions of the lowest Feynman diagrams only. It implies that in some sense the coupling constant should be small enough. The Bethe-Salpeter equation is the most important integral equation of this type and it is widely used, especially for calculation of relativistic corrections in hydrogen-like systems and deuteron physics (see, for example, [1,3,7,9]).

Now we would like to pay attention to the so-called $Z_2 = 0$ approach (see, for example, [8,10]) which is not used so widely, although the bosonization of QCD introducing bilocal boson-type fields was developed in [11]. The objection and prejudice against this method are based on the persuasion that in local quantum field theory we should have local interaction only. For example, the pion as a quark-antiquark bound state is represented by the local term $\pi(\bar{q}\gamma_5 q)$. As a result the renormalization constant Z_2 for the pion contains the ultraviolet divergence and therefore the condition $Z_2 = 0$ makes no practical sense. The answer is that the local vertex *does not lead* to any bound state. The vertex connected with the formation of a bound state *should be nonlocal*. Moreover, this nonlocal vertex in the nonrelativistic limit is directly connected with the nonrelativistic wave function of this bound state. The question arises how to find this nonlocal vertex. In the approach [11] this vertex is a solution of the Bethe-Salpeter equation, which can only be solved by difficult numerical computations.

Recently, in papers [14] we began to develop *the Model of Induced Quark Currents* based on the assumpsion that (anti-)self-dual homogeneous gluon field realizes the physical QCD vacuum. To calculate the mass spectrum of bosons as quark-antiquark bound states, the method of so-called *Bosonization of Nonlocal Currents* (BNC) was used. Numerical results for the boson spectrum turn out to be in good agreement with the experimental ones. This method is quite close to the $Z_2 = 0$ method and to the bosonization of QCD introduced by [11]. A similar approach was developed for fermionic many-body problems in statistical physics in [12]. The idea of BNC consists of two point. First, we write the Green function containing a bound state in the functional integral representation in which the Gaussian measure being the lowest order of the bosonization of bilocal currents contains the Bethe-Salpeter kernel written in symmetric hermitian form. Second, we have guessed the analytical form of the orthonormal system of functions very close to the eigenfunctions of the Bethe-Salpeter equation.

Now we would like to attract attention to the following quite important point which is not usually stressed clearly enough. Because exact solutions for any realistic quantum field model are not known one or another approximation should be used. In any case, we should have possibilities, at least in principle, to evaluate the theoretical accuracy of a chosen approximation. The most simple, although maybe not the most accurate, way is to require the effective coupling constant to be noticeably smaller than unity. The perturbation series in the coupling constant is the most popular method of calculations, especially in quantum electrodynamics or the theory of weak interactions. However, if a perturbation series is rearranged by a subsummation, the problem of the effective coupling constant becomes quite topical. Thus, if the theory is reformulated somehow, the effective coupling constant should be defined for each particular case and should be smaller than one. Only in this case we can trust our computations.

The aim of this paper is to formulate the BNC-method and to clarify the connection between the principal approaches used in atomic, nuclear and particle physics of today to calculate bound states of interacting particles, namely, the nonrelativistic potential Schrödinger equation (S-method), the relativistic Bethe-Salpeter equation in the one-boson-exchange approximation (the BS-method) and our BNC-method in the one-loop approximation. We answer this question by considering a relatively simple quantum field model. The results will provide a deeper understanding of approximations used in the well-known standard approaches.

An example of the above-mentioned simple quantum field model is the Yukawa interaction of charged scalar bosons described by the field Φ and neutral bosons described by the field ϕ. The Lagrangian density is

$$L(x) = \Phi^+(\Box - M^2)\Phi + \frac{1}{2}\phi(\Box - m^2)\phi + g\Phi^+\Phi\phi. \tag{1}$$

This model is frequently used as the simplest pattern of QFT in many discussions, although this system is not stable from a strict point of view because the Hamiltonian is not bounded ¿from below. This model has been investigated by various methods (see, for example, [1,7,15] and the recent paper [13] and references therein).

In this model it is possible to retrace all details of bound states arising in quantum field theory. Generalization to the case of the Dirac field presents no difficulties of principle and leads to technical problems connected with the algebra of γ-matrices only. This model is superrenormalizable so that the renormalization procedure has the simplest form. The main aim of this paper is to understand the general mechanism of bound states arising in this quantum field model and to clarify conditions on parameters of this quantum field model dictating a definite method mentioned above to be applied.

The model contains three dimensionless parameters:

$$\lambda = \frac{g^2}{16\pi M^2}, \quad \xi = \frac{m}{M}, \quad b = \left(\frac{M_b}{2M}\right)^2 = \left(1 - \frac{\Delta M}{2M}\right)^2. \tag{2}$$

where $M_b = 2M - \Delta M$ is the mass of a supposed bound state, ΔM is the mass excess or the binding energy.

Three parameters, λ, ξ and b, are not independent. The standard formulation of the problem is to find b if ξ and λ are given. We formulate the problem in another way: what is the region of changing b for a fixed ξ, if the effective coupling constant (which can differ ¿from λ) is smaller than unity?

The parameter ξ is supposed to be smaller than 1. Our aim is to find the condition under which the mass M_b of a bound state lies in the interval

$$0 < M_b < 2M \qquad \text{or} \qquad 0 < b = \left(\frac{M_b}{2M}\right)^2 < 1,$$

i.e. this bound state should be stable.

It turns out that all approaches mentioned above require the effective coupling constant to be small enough. The value of the parameter $\xi = \frac{m}{M}$ plays the crucial role separating nonrelativistic and relativistic approaches. Namely

- the *Potential nonrelativistic picture* takes place for

$$\xi = \frac{m}{M} \ll 1, \qquad \lambda = \frac{g^2}{16\pi M^2} \ll 1, \tag{3}$$

  and, therefore,

$$1 - b = \left[1 - \left(\frac{M_b}{2M}\right)^2\right] \ll 1, \qquad \text{or} \qquad \Delta M \ll 2M,$$

  i.e., the binding energy is very small;

- the *Bethe-Salpeter approach and Bosonization of Nonlocal Currents* take place for

$$\xi = \frac{m}{M} \leq 1, \qquad \lambda = \frac{g^2}{16\pi M^2} \leq 1, \tag{4}$$

  and the binding energy can be up to $\Delta M \leq (.2 \div .3) \cdot 2M$.

The result of this paper is that all methods under consideration can be used in the weak coupling regime only, but the nonrelativistic potential picture takes place provided the mass of an exchange particle m to be very small in comparison with that of the constituent particle M. The BS-approach and the Bosonization of Nonlocal Currents can be used for any relations between the masses of exchange and constituent particles although the latter method has a somewhat wider range of applicability.

From my point of view, it is just the BNC-formalism to be the most attractive method to study bound states in QFT. It provides to represent results in an analytic form and to evaluate the theoretical accuracy of approximations.

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II. THE FORMULATION OF THE PROBLEM

We consider the interaction of scalar charged scalar fields $\Phi(x)$ and a neutral scalar field $\phi(x)$. All the consideration is given in the Euclidean metrics. The total Lagrangian is done by (1). The propagator of a scalar particle with the mass M is

$$D_M(x-y) = \int \frac{dp}{(2\pi)^4} \cdot \frac{e^{ip(x-y)}}{M^2 + p^2}. \tag{5}$$

The object of our interest is the four-point Green function

$$\begin{aligned} &G(x_1, x_2, y_1, y_2) \\ &= \int\int D\Phi D\Phi^+ D\phi \cdot \Phi^+(x_1)\Phi(x_2)\Phi^+(y_1)\Phi(y_2) \cdot e^{S[\Phi^+,\Phi,\phi]}, \\ &\qquad S[\Phi^+, \Phi, \phi] = \int dx L(x), \end{aligned} \tag{6}$$

where an appropriate normalization should be introduced.

The four-point Green function (6) contains all information about possible bound states in the channels $\Phi\Phi$ and $\Phi^+\Phi$. The particles Φ can be called constituent particles. We take an interest in bound states in the channel $\Phi^+\Phi$. The quantum numbers Q can be fixed by an appropriate vertex

$$\Phi^+(x_1)\Phi(x_2) \to J_Q(x) = (\Phi^+ V_Q \Phi)_x = \Phi^+(x) V_Q(\overleftrightarrow{p}_x)\Phi(x),$$
$$\overleftrightarrow{p}_x = \frac{1}{i}\left[\overleftarrow{\partial}_x - \overrightarrow{\partial}_x\right].$$

The nonlocal vertex $V_Q(\overleftrightarrow{p}_x)$ defines the quantum numbers Q of the current $J_Q = (\Phi^+ V_Q \Phi)$. This vertex can be represented like

$$V_Q(\overleftrightarrow{p}_x) = \int du\ \tilde{V}_Q(u) e^{iu\overleftrightarrow{p}_x}, \tag{7}$$

The current $J_Q = (\Phi^+ V_Q \Phi)$ can be written as

$$J_Q(x) = (\Phi^+ V_Q \Phi)_x = (\Phi^+(x) V_Q(\overleftrightarrow{p}_x)\Phi(x)) \tag{8}$$
$$= \int du\ \Phi^+(x+u)\tilde{V}_Q(u)\Phi(x-u).$$

The Green function with quantum numbers Q is defined as

$$G_Q(x-y) = \int D\Phi D\Phi^+ \int D\phi\ J_Q(x) J_Q(y)\ e^{S[\Phi^+,\Phi,\phi]}. \tag{9}$$

If in this channel a stable bound state with the mass $M_Q = M_b < 2M$ does exist, the Green function $G_Q(x)$ has the following asymptotic behaviour

$$G_Q(x) \sim e^{-M_Q|x|} \qquad \text{for} \qquad |x| \to \infty,$$

so that the mass of the state $J_Q = (\Phi^+ V_Q \Phi)$ can be found as

$$M_b = M_Q = -\lim_{x\to\infty} \frac{1}{|x|} \ln G_Q(x). \tag{10}$$

The problem is to calculate the functional integral in representation (9) and find the mass M_b according to (10).

If we consider perturbation expansion over the coupling constant g for the four point Green function $G(x_1, x_2, y_1, y_2)$, we will get a series of the Feynman diagrams describing an interaction of two particles Φ. This series can be written in the form of the Bethe-Salpeter equation (see, for example, [1,9,3]). Bound states of two particles Φ in a channel $J_Q(x) = (\Phi^+ V_Q \Phi)_x$ can be found as solutions of this equation.

We proceed in another way. First, our aim is to obtain for the Green function (9) the functional integral representation in which the term being responsible for bound state creation would be written in the explicit form and for remaining corrections the smallness criterion would be defined. Second, we want to formulate the method of analytical calculations of binding energies and evaluate their theoretical accuracy.

Fortunately, it is possible in the representation (9) to do the first integration either over the field $\Phi(x)$ or field $\phi(x)$. Thus, we get two representations which are the starting points of two approaches: the *Potential picture* and the *Bosonization of Nonlocal Currents*.

I. Potential picture

The integration in (6) over the charged scalar field Φ gives for the Green function (9):

$$G_Q(x-y) = G_Q^{(P)}(x-y) + G_Q^{(A)}(x-y). \tag{11}$$

Here

$$G_Q^{(P)}(x-y) = \int du \int dv \int D\phi\ e^{S_P[\phi]}$$
$$\cdot \left\{ \tilde{V}_Q(u) \mathbf{D}_M(x+u, y+v|\phi) \tilde{V}_Q(v) \mathbf{D}_M(y-v, x-u|\phi) \right\},$$
$$G_Q^{(A)}(x-y) = \int du \int dv \int D\phi\ e^{S_P[\phi]}$$
$$\cdot \left\{ \tilde{V}_Q(u) \mathbf{D}_M(x+u, x-u|\phi) \right\} \cdot \left\{ \tilde{V}_Q(v) \mathbf{D}_M(y+v, y-v|\phi) \right\},$$

where

$$S_P[\phi] = \frac{1}{2} \int dx\ \phi(x)(\Box - m^2)\phi(x) - \mathrm{tr} \ln[1 - g\phi D_M].$$

The Green function $\mathbf{D}_M(x, y|\phi)$ satisfies the equation

$$[-\Box + M^2 - g\phi(x)]\mathbf{D}_M(x, y|\phi) = \delta(x-y) \tag{12}$$

with

$$\mathbf{D}_M^+(x, y|\phi) = \mathbf{D}_M(x, y|\phi) = \mathbf{D}_M(y, x|\phi).$$

Here the functions $G_Q^{(P)}(x)$ and $G_Q^{(A)}(x)$ are said to be "potential" and "annihilation" Green functions, respectively. The approach based on the representation (11) will be called the *Potential picture*.

II. Bosonization of Nonlocal Currents

The integration in (9) over the scalar field $\phi(x)$ gives

$$G_Q(x-y) = \int \int D\Phi D\Phi^+ \cdot e^{S_B[\Phi^+, \Phi]} \cdot (\Phi^+ V_Q \Phi)_x (\Phi^+ V_Q \Phi)_y, \tag{13}$$

where an appropriate normalization is implied and

$$S_B[\Phi^+, \Phi] = -(\Phi^+ D_M^{-1} \Phi) + \frac{g^2}{2}(\Phi^+ \Phi D_m \Phi^+ \Phi),$$
$$(\Phi^+ D_M^{-1} \Phi) = \int dx\ \Phi^+(x)(-\Box + M^2)\Phi(x),$$
$$\frac{g^2}{2}(\Phi^+ \Phi D_m \Phi^+ \Phi) = \int dx \int dy\ \Phi^+(x)\Phi(x) D_m(x-y) \Phi^+(y)\Phi(y).$$

The approach based on the representation (13) will be called *the Bosonization of Nonlocal Currents*.

III. THE POTENTIAL PICTURE.

The starting point of the Potential picture is the representation (11).

A. The Green function $\mathbf{D}_M(x, y|\phi)$

First, the charged loops should be neglected, so that the Green functions $G_P(x-y)$ and $G_Q(x-y)$ are represented by

$$G^{(P)}(x-y) = \int d\sigma_{uv}[\phi]\ \mathbf{D}_M(x+u, y+v|\phi) \cdot \mathbf{D}_M(x-u, y-v|\phi), \tag{14}$$
$$G^{(A)}(x-y) = \int d\sigma_{uv}[\phi]\ \mathbf{D}_M(x+u, x-u|\phi) \cdot \mathbf{D}_M(y+v, y-v|\phi)$$
$$d\sigma_{uv}[\phi] = du\tilde{V}(u)\ dv\tilde{V}(v)\ D\phi\ e^{-\frac{1}{2}(\phi(x) D_m^{-1} \phi)}.$$

We would like to stress now that the neglect of loops of scalar particles Φ presupposes the dimensionless coupling constant λ to be small enough.

The solution of (12) can be represented by the functional integral (see, for example, [16]):

$$\mathbf{D}_M(x,y|\phi) = \frac{1}{-\Box + M^2 - g\phi(x)} \cdot \delta(x-y) \tag{15}$$
$$= \int\limits_0^\infty \frac{d\alpha}{8\pi^2\alpha^2} e^{-\frac{\alpha}{2}M^2} \int D\xi \exp\left\{ -\int\limits_0^\alpha d\tau \frac{\dot\xi^2(\tau)}{2} - \frac{g}{2}\int\limits_0^\alpha d\tau \phi\left(\xi(\tau)\right)\right\},$$

with the boundary conditions $\xi(0) = y, \;\; \xi(\alpha) = x$ and the normalization

$$\int D\xi \exp\left\{ -\int\limits_0^\alpha d\tau \frac{\dot\xi^2(\tau)}{2}\right\} = e^{-\frac{(x-y)^2}{2\alpha}}.$$

B. The Green function $G^{(P)}(x)$

The function $G^{(P)}(x)$ after integration over ϕ has the form

$$G_P(x) = \left(\frac{M}{8\pi^2 x}\right)^2 \int d\Sigma_1 d\Sigma_2 \; e^{W_{11}+2W_{12}+W_{22}},$$

$$W_{ij} = \frac{g^2}{8}\int\limits_0^{\alpha_i} d\tau_1 \int\limits_0^{\alpha_j} d\tau_2 \; D_m\left(\xi_i(\tau_1) - \xi_j(\tau_2)\right),$$

$$\int d\Sigma_j\{*\} = \int du_j \tilde V(u_j) \int\limits_0^\infty d\alpha_j \; e^{-\frac{\alpha_j}{2}M^2} \int D\xi_j \exp\left\{ -\int\limits_0^{\alpha_j} d\tau \frac{\dot\xi_j^2(\tau)}{2}\right\}\{*\},$$

$$(j = 1,2; \quad u_1 = u, \;\; u_2 = v),$$

$$\xi_1(0) = v, \qquad \xi_1(\alpha_1) = x + u, \qquad \xi_2(0) = -v, \qquad \xi_2(\alpha_2) = x - u.$$

where for simplicity we put $y = 0$.

Our task is to get the asymptotic behaviour of the functions $G^{(P)}(x)$ for asymptotically large $x = \sqrt{x^2} \to \infty$. To this end, let us introduce the following variables:

$$\alpha_j = \frac{x}{M s_j}, \qquad \tau_j = \frac{\beta_j}{M s_j},$$

$$\xi_1(\beta) = n\beta + \eta_1(\beta), \qquad \xi_2(\beta) = n\beta + \eta_2(\beta), \qquad n_\mu = \frac{x_\mu}{x}.$$

Then, one can obtain

$$G^{(P)}(x) = \left(\frac{M}{8\pi^2 x}\right)^2 \int du \; \tilde V(u) \int dv \; \tilde V(v) \tag{16}$$
$$\cdot \int\limits_0^\infty \int\limits_0^\infty ds_1 ds_2 \; e^{-\frac{xM}{2}\left(\frac{1}{s_1} + s_1 + \frac{1}{s_2} + s_2\right)} \cdot J^{(P)}(s_1, s_2; x),$$

where

$$J^{(P)}(s_1,s_2;x) \tag{17}$$
$$= \int\int D\eta_1 D\eta_2 \exp\left\{ -\int\limits_0^x d\beta \left[\frac{Ms_1\dot{\eta}_1^2(\beta)}{2} + \frac{Ms_2\dot{\eta}_2^2(\beta)}{2} \right] + W_x[\eta_1,\eta_2] \right\},$$
$$\eta_1(0) = v, \quad \eta_1(x) = u, \qquad \eta_2(0) = -v, \quad \eta_2(x) = -u.$$

The "potential interaction" is described by the two-point nonlocal functional

$$W_x[\eta_1,\eta_2] = W_{11} + 2W_{12} + W_{22}, \tag{18}$$
$$W_{ij} = \frac{g^2}{8M^2 s_i s_j} \int\limits_0^x \int\limits_0^x d\beta_1 d\beta_2 \; D_m\left(n(\beta_1-\beta_2) + \eta_i(\beta_1) - \eta_j(\beta_2)\right).$$

The functional integral for $J_x(s_1,s_2;x)$ looks like the Feynman path integral in the nonrelativistic statistic quantum mechanics for the four dimensional motion of particles $\eta_1(\beta)$ and $\eta_2(\beta)$ with "masses" Ms_1 and Ms_2 where β plays the role of the imaginary time or temperature. The interaction of these particles is defined by the nonlocal functional $W_{11}+2W_{12}+W_{22}$ which contains potential W_{12} and nonpotential $W_{11}+W_{22}$ interactions. It should be noted that the structure of the functional integral (17) reminds the polaron problem (see [17]).

The asymptotic form of the function $J^{(P)}(s_1,s_2;x)$ looks like

$$J^{(P)}(s_1,s_2;x) \sim \exp\{-xE(s_1,s_2)\}, \tag{19}$$

where $E(s_1,s_2)$ is the energy of the lowest bound state. The asymptotic behaviour of the functional $G^{(P)}(x)$ as $x\to\infty$ is determined by the saddle point of the integrals over s_1 and s_2 in the representation (16). Substituting expression (19) into (16), one can get

$$M_b = -\lim_{x\to\infty} \frac{1}{|x|} \ln G^{(P)}(x), \tag{20}$$
$$= \min_{(s_1,s_2)} \left[\frac{M}{2}\left(\frac{1}{s_1} + s_1 + \frac{1}{s_2} + s_2 \right) + E(s_1,s_2) \right]$$
$$= \min_s \left[M\left(\frac{1}{s} + s \right) + E(s,s) \right].$$

The main problem is to compute the functional integral (17). This computation can be done by the variational methods (see [17]). We plan to calculate this functional integral applying the Gaussian equivalent representation method which was sucessfully used sucessfully for the polaron problem (see [16]).

Besides one can see that the representations (16) and (17) does not really feel the explicit form of the vertex $\tilde{V}_Q(u)$ although it should extract a bound state with definite quantum numbers Q. It means, in fact, that in the general case, i.e., for any value of the coupling constant g, the modern analytical methods, applied to the functional integral (16), allow to calculate with reasonable accuracy the energy of the lowest bound state only.

C. The Green function $G^{(A)}(x-y)$

The function $G^{(A)}(x)$ after integration over ϕ has the form

$$G^{(A)}(x) = \left(\frac{M}{8\pi^2 x} \right)^2 \int d\Sigma_1 d\Sigma_2 e^{W_{11}+2W_{12}+W_{22}},$$

$$W_{jj} = \frac{g^2}{2} \int\limits_0^{\alpha_j} d\tau_1 \int\limits_0^{\alpha_j} d\tau_2 D_m\left(\xi_j(\tau_1) - \xi_j(\tau_2)\right), \qquad (j=1,2),$$

$$W_{12} = \frac{g^2}{2} \int\limits_0^{\alpha_1} d\tau_1 \int\limits_0^{\alpha_2} d\tau_2 D_m \left(x + \xi_1(\tau_1) - \xi_2(\tau_2) \right).$$

Here $\xi_1(0) = -u, \qquad \xi_1(\alpha_1) = u, \qquad \xi_2(0) = -v, \qquad \xi_2(\alpha_2) = v.$

One can see that in the limit $x \to \infty$ $W_{12} \to 0$ and

$$G^{(A)}(x) \to \left[\frac{M}{8\pi^2 x} \int d\Sigma_1 e^{W_{11}} \right]^2,$$

so that no bound state arises in this case. Thus, the annihilation channel does not contain any bound states. In other words, intermediate pure boson states of particles ϕ cannot produce any bound state.

D. The Nonrelativistic Limit

In this section we obtain the nonrelativistic limit $c \to \infty$ for the loop function $G_P(x)$ in (16). Our task is to introduce the parameter c in an explicit form into $G_P(x)$ and then go to the limit $c \to \infty$. To this end, let us restore the parameter c in our formulas. We have

$$M \to Mc, \qquad g \to g, \tag{21}$$

$$x_\mu = (x_4, \mathbf{x}) \to (ct, \mathbf{x}), \qquad x = \sqrt{x^2} = \sqrt{c^2 t^2 + \mathbf{x}^2} \to ct,$$

$$n_\mu = \frac{x_\mu}{\sqrt{x^2}} \to (1, \mathbf{x}/ct) \to (1, 0).$$

The propagator $D_m(x)$ becomes

$$\begin{aligned} D_m(x) &\to c D_\kappa(ct, \mathbf{x}) \qquad (22) \\ &= c \int \frac{d^4k}{(2\pi)^4} \tilde{D}_m(k^2) e^{i(k_4 t + \mathbf{kx})} = \int \frac{d\mathbf{k}}{(2\pi)^3} \int\limits_{-\infty}^{\infty} \frac{dv}{2\pi} \cdot \frac{e^{i(vt + \mathbf{kx})}}{\mathbf{k}^2 + \kappa^2 + \frac{v^2}{c^2}} \\ &\to \int \frac{d\mathbf{k}}{(2\pi)^3} \int\limits_{-\infty}^{\infty} \frac{dv}{2\pi} \cdot \frac{e^{i(vt + \mathbf{kx})}}{\mathbf{k}^2 + \kappa^2} \cdot \left[1 - \frac{v^2}{c^2} \cdot \frac{1}{\mathbf{k}^2 + \kappa^2} \right] \\ &= \delta(t) \cdot \frac{1}{4\pi} \frac{e^{-\kappa r}}{r} + \frac{1}{c^2} \delta''(t) \cdot \frac{1}{8\pi} \frac{e^{-\kappa r}}{\kappa} + O\left(\frac{1}{c^4} \right). \end{aligned}$$

The parameter $\frac{1}{\kappa}$ defines the radius of the nonrelativistic Yukawa potential, therefore, we have to keep

$$\kappa = mc = \text{const}$$

to be finite in the limit $c \to \infty$.

Let us come back to the Green function (16). According to (21) it reads

$$G_P(t) = \left(\frac{M}{4\pi t} \right)^2 \iint\limits_0^\infty ds_1 ds_2 \; e^{-t \frac{Mc^2}{2} \left(\frac{1}{s_1} + s_1 + \frac{1}{s_2} + s_2 \right)} J_P(s_1, s_2; t)$$

In the functional integral (17) let us introduce the new varibles $\beta_j = c\tau_j$. It is convinient to represent the four-vectors η_j in the one and three component form

$$\eta_j(\beta) = (a_j(\tau), \mathbf{b}_j(\tau)), \qquad (j = 1, 2)$$

and

$$a_1(\tau) = \Phi(\tau) + \frac{\phi(\tau)}{2}, \qquad a_2(\tau) = \Phi(\tau) - \frac{\phi(\tau)}{2},$$

$$\mathbf{b}_1(\tau) = \mathbf{R}(\tau) + \frac{\mathbf{r}(\tau)}{2}, \qquad \mathbf{b}_2(\tau) = \mathbf{R}(\tau) - \frac{\mathbf{r}(\tau)}{2}.$$

with the boundary conditions

$$\mathbf{R}(0) = \mathbf{R}(t) = 0, \qquad \mathbf{r}(0) = 2\mathbf{u}, \quad \mathbf{r}(t) = 2\mathbf{v}$$

In the limit $c \to \infty$ the integral for $G_P(t)$ over s_1 and s_2 can be calculated by the saddle-point method. The saddle points are $s_1 = s_2 = 1$ and we get

$$G_P(t) \approx \frac{M}{8\pi t^3 c^2} e^{-2Mc^2 t} \cdot J_P(t) \tag{23}$$

where the functional integral looks

$$J_P(t) = \iint Da_1 Da_2 \iint D\mathbf{b}_1 D\mathbf{b}_2 \; e^{-\frac{M}{2}\int\limits_0^t d\tau \left[\dot{a}_1^2 + \dot{a}_2^2 + \dot{\mathbf{b}}_1^2 + \dot{\mathbf{b}}_2^2\right] + W_{11} + 2W_{12} + W_{22}}$$

$$= \iint D\Phi D\phi \iint D\mathbf{R} D\mathbf{r} \; e^{-M\int\limits_0^t d\tau \left[\dot{\Phi}^2 + \frac{1}{4}\dot{\phi}^2 + \dot{\mathbf{R}}^2 + \frac{1}{4}\dot{\mathbf{r}}^2\right] + W_{11} + 2W_{12} + W_{22}}.$$

The functionals W_{11} and W_{22} describe the mass renormalization in the nonrelativistic limit and we omit them. According to (21) and (22) the functional W_{12} (18), where the argument of D-function is

$$n(\beta_1 - \beta_2) + \eta_1(\beta) - \eta_2(\beta) = (c(\tau_1 - \tau_2) + a_1(\tau_1) - a_2(\tau_2), \mathbf{b}_1(\tau_1) - \mathbf{b}_2(\tau_2)),$$

acquires the form in the limit $c \to \infty$

$$\begin{aligned} 2W_{12} &= \frac{g^2}{16\pi M^2} \iint\limits_0^t d\tau_1 d\tau_2 \left\{ \left[\delta(\tau_1 - \tau_2) + \frac{1}{c}\delta'(\tau_1 - \tau_2)(a_1(\tau_1) - a_2(\tau_2)) \right] \cdot \frac{e^{-\kappa r}}{r} \right. \\ &+ \left. \frac{1}{2c^2}\delta''(\tau_1 - \tau_2) \cdot \frac{e^{-\kappa r}}{\kappa} + ... \right\} \\ &= \lambda \int\limits_0^t d\tau \; \left\{ \left[1 + \frac{1}{8c^2} \left(\dot{\mathbf{r}}^2 - \frac{(\dot{\mathbf{r}}\mathbf{r})^2}{r^2}(1 + \kappa r) \right) \right] \cdot \frac{e^{-\kappa r}}{r} - \frac{1}{c}\dot{\Phi}\frac{e^{-\kappa r}}{r} \right. \\ &+ \left. O\left(\frac{1}{c}\phi\dot{\mathbf{R}}, \frac{1}{c^2}\dot{\mathbf{R}}^2 \right) \right\} \end{aligned}$$

where $r = r(\tau) = |\mathbf{b}_1(\tau) - \mathbf{b}_2(\tau)|$.

After integration over Φ, ϕ and $\mathbf{R}$ in the functional integral for J_P one can get

$$J_P(t) = \int\limits_{\mathbf{r}(0)=2\mathbf{u}}^{\mathbf{r}(t)=2\mathbf{v}} D\mathbf{r} \; \cdot \exp\left\{ -\int\limits_0^t d\tau \; \left[\frac{M_c}{2}\dot{\mathbf{r}}^2(\tau) + U(\mathbf{r}(\tau)) \right] \right\}, \tag{24}$$

$$U(\mathbf{r})) = -\lambda\frac{e^{-\kappa r}}{r} - \frac{\lambda}{2M_c^2 c^2} p_i \left(\left[\delta_{ij} - \frac{r_i r_j}{r^2}(1 + \kappa r) \right] \cdot \frac{e^{-\kappa r}}{r} \right) p_j - \frac{\lambda^2}{8M_c c^2}\frac{e^{-2\kappa r}}{r^2} \tag{25}$$

where $M_c = \frac{M}{2}$ and $p_j = M_c \dot{r}_j$.

Here we keep the main terms contributing to the potential and neglect terms describing and the nonlocal interaction and terms of the order $O(1/c^4)$.

One can see that this representation for $J_P(t)$ coincides with the Feynman path integral in quantum machanics for the Green function $K(\mathbf{v}, t; \mathbf{u}, 0)$

$$K(\mathbf{v},t;\mathbf{u},0) = \int\limits_{\mathbf{r}(0)=2\mathbf{u}}^{\mathbf{r}(t)=2\mathbf{v}} D\mathbf{r} \exp\left\{ -\int\limits_0^t d\tau \left[\frac{M_{cm}}{2}\dot{\mathbf{r}}^2(\tau) + U(\mathbf{r}(\tau)) \right] \right\}, \tag{26}$$
$$= \sum_Q \psi_Q(\vec{v}) e^{-tE_Q} \psi_Q(\vec{u})$$

Here $\psi_Q(\mathbf{r})$ and E_Q are eigenfunctions and eigenvalues for the quantun number Q connected with the space $\mathbf{R}^3$ of the Schrödinger equation

$$\left[\frac{p^2}{2M_{cm}} + U(\mathbf{r}) \right] \psi_Q(\vec{r}) = E_Q \psi_Q(\mathbf{r}) \tag{27}$$

where $U(\mathbf{r}) < 0$ is the attractive potential.

As a result, the Green function $G_P(t)$ for $t \to \infty$ behaves like

$$G_P(t) = \sum_Q e^{-tE_Q} \left[\int d\mathbf{u}\ \tilde{V}(\mathbf{u}) \psi_Q(\mathbf{u}) \right]^2.$$

If we choose

$$\tilde{V}(\mathbf{u}) = \int du_4\ \tilde{V}(u_4, \mathbf{u}) = \psi_{Q_0}(\mathbf{u}),$$

then

$$\int d\mathbf{u}\ \psi_Q(\mathbf{u}) \psi_{Q_0}(\mathbf{u}) = \delta_{QQ_0},$$

and, finally, for large t we have

$$G_P(t) \to e^{-tE_{Q_0}},$$

where E_{Q_0} is the energy of the bound state of two nonrelativistic particles in the quantum state Q_0 arising due to the potential $U(\mathbf{r})$. The mass of the bound state in the nonrelativistic approach is

$$M_{Q_0} = \frac{1}{c^2}\left[Mc^2 + E_{Q_0} + O\left(\frac{1}{c}\right) \right] = 2M + \frac{E_{Q_0}}{c^2} + O\left(\frac{1}{c^3}\right). \tag{28}$$

Thus, in the nonrelativitic limit the relativistic vertex $\tilde{V}(u)$ is connected with the nonrelativistic eigenfunction

$$\tilde{V}(u) \to \int du_4\ \tilde{V}(u_4, \mathbf{u}) = \tilde{V}(\mathbf{u}) = \psi_Q(\mathbf{u}) \tag{29}$$

and the bound state mass of two scalar particles is a sum of their masses plus the binding energy E_Q determined by the nonrelativistic potential interaction.

E. The nonrelativistic Yukawa potential and relativistic corrections.

Thus in the nonrelativistic limit, we have the Schrödinger equation with relativistic corrections:

$$H = H_Y + \frac{1}{c^2}(U_p + U_s) + O\left(\frac{1}{c^4}\right) \tag{30}$$

where H_Y is the Yukawa potential

$$H_Y = \frac{p^2}{2M_{cm}} - \lambda\frac{e^{-mr}}{r}, \qquad M_{cm} = \frac{M}{2}, \qquad m = \kappa,$$

and U_p and U_s are the first and second lowest relativistic corrections in (24).

Our problem is to find restrictions on the parameters λ and $\xi = \frac{m}{M}$ for which the ralativistic corrections U_p and U_s can be neglected. We proceed in the simplest way. We use the variation function

$$\Psi(r) = e^{-\frac{1}{2}msr},$$

where s is the variational parameter, for the ground state. This approach gives us quite a good qualitative and even semi-quantitative estimation of the background energy. It is sufficient for our aim. The calculations give

$$\begin{aligned} E_Y &\approx \min_a \frac{\langle\Psi|H_Y|\Psi\rangle}{\langle\Psi|\Psi\rangle} = \frac{m^2}{4M} \cdot \min_s \left\{ s^2 - G \cdot \frac{s^3}{(1+s)^2} \right\} \\ &= -\frac{m^2}{8M_c} \cdot \frac{s^2(s-1)}{3+s}, \end{aligned}$$

where the parameter s is defined by the equation

$$G = 4\lambda \cdot \frac{M_c}{m} = \frac{2(1+s)^3}{s(3+s)}.$$

The bound state can exist if $s > 1$, i.e.

$$G > 4, \qquad \text{or} \qquad \lambda > \frac{m}{M_c}$$

The contribution of relativistic corrections can be evaluated as follows:

$$\begin{aligned} \delta_p &= \frac{\langle\Psi|U_p|\Psi\rangle}{\langle\Psi|U_Y|\Psi\rangle} = \frac{1}{4} \cdot \left(\frac{m}{M}\right)^2 \cdot \frac{s^2}{1+s}, \\ \delta_s &= \frac{\langle\Psi|U_s|\Psi\rangle}{\langle\Psi|U_Y|\Psi\rangle} = \frac{\lambda}{8} \cdot \frac{m}{M} \cdot \frac{(1+s)^2}{2+s}, \end{aligned}$$

Thus the nonrelativistic picture takes place if

$$|E_Y| \ll 2M, \qquad \delta_p \ll 1, \qquad \delta_s \ll 1$$

It is easy to see that these inequalities take place if

$$\xi = \frac{m}{M} \ll 1, \qquad \lambda \ll 1. \tag{31}$$

It is valid for $s \sim 2 \div 3$ and $a \sim (1 \div 2) \cdot m$.

F. Relativistic incompleteness of quantum mechanics of two particles.

Here we would like to pay attention to the Schrödinger equation which describes two nonrelativistic particles

$$H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} - U(\vec{r}_1 - \vec{r}_2),$$

where the potential is attractive.

Let us pass to the center-of-mass system in a standard way

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}, \qquad \vec{r} = \vec{r}_1 - \vec{r}_2.$$

The Hamiltonian takes the form

$$H = \frac{p^2}{2M} + \frac{p_r^2}{2\mu} - U(r)$$

$$M = m_1 + m_2, \qquad \mu = \frac{m_1 m_2}{m_1 + m_2}$$

The solution of the Schrödinger equation $H\Psi = E\Psi$ can be written as

$$\Psi(\vec{R},\vec{r}) = e^{i\vec{p}\vec{R}}\psi(r)$$

where $\vec{p}$ is the momentum of the total system; $\psi(r)$ is an eigenfunction of the equation

$$\left[\frac{\vec{p}_r^2}{2\mu} - U(r)\right]\psi(r) = -\varepsilon\psi(r),$$

and $-\varepsilon$ ($\varepsilon > 0$) is an eigenvalue of a bound state. Then, the eigenvalue or the energy of the state $\Psi(\vec{R},\vec{r})$ for $\vec{p} \neq 0$ is

$$E = \frac{p^2}{2M} - \varepsilon.$$

From the physical point of view this energy has no reasonable meaning. Indeed, we should get

$$E = \frac{p^2}{2M} - \varepsilon \to \frac{p^2}{2M_{phys}},$$
$$M_{phys} = m_1 + m_2 - \Delta, \qquad \Delta = \frac{\varepsilon}{c^2}$$

i.e., the interaction between two particles should give the mass excess.

On the other hand, the latter formula can be obtained from the relativistic energy in the nonrelativistic limit

$$\begin{aligned} E &= \sqrt{M_{phys}^2c^4 + p^2c^2} = M_{phys}c^2 + \frac{p^2}{2M_{phys}} + O(p^2) \\ &= (m_1 + m_2 - \Delta M)c^2 + \frac{p^2}{(m_1 + m_2 - \Delta M)} + O(p^2) \\ &= (m_1 + m_2)c^2 + \frac{p^2}{2(m_1 + m_2)} - \Delta Mc^2 + O\left(\frac{\Delta M}{M}\right) + O(p^2) \end{aligned}$$

and the mass of the bound state equals

$$M_{phys} = M - \frac{\varepsilon}{c^2} + O\left(\frac{1}{c^4}\right).$$

Thus, the nonrelativistic Schrödinger equation describing two nonrelativistic particles can be cosidered as a fragment of an relativistic equation describing the relativistic interaction of two particles. The "true" relativistic equation should contain terms describing both the motion of the center of masses and their relative motion.

IV. BOSONIZATION OF NONLOCAL CURRENTS.

The starting point of *Bosonization of charged currents* is the representation (13).

A. Bilocal currents.

Let us consider the four-field term (13) and introduce the bilocal currents:

$$\begin{aligned} &\frac{g^2}{2}(\Phi^+\Phi D_m\Phi^+\Phi) \\ &= \frac{g^2}{2}\int\int dy_1dy_2\ \Phi^+(y_1)\Phi(y_1)D_m(y_1 - y_2)\Phi^+(y_2)\Phi(y_2) \\ &= \frac{g^2}{2}\int\int dy_1dy_2\ J^+(y_1,y_2)J(y_1,y_2) = \frac{g^2}{2}(J^+J) \end{aligned}$$

where

$$J(y_1,y_2)=\sqrt{D_m(y_1-y_2)}(\Phi^+(y_1)\Phi(y_2)),\qquad J^+(y_1,y_2)=J(y_2,y_1). \tag{32}$$

The next step is to use the Gaussian representation

$$e^{\frac{g^2}{2}(\Phi^+\Phi D_m\Phi^+\Phi)}=e^{\frac{g^2}{2}(J^+J)}=\int DA\ e^{-\frac{1}{2}(A^+A)-g\ (A^+J)}.$$

where $A^+(y_1,y_2)=A(y_2,y_1)$ and

$$\begin{aligned}DA&=\prod_{y_1,y_2}dA(y_1,y_2),\\ (A^+A)&=\int\int dy_1dy_2\ A^+(y_1,y_2)A(y_1,y_2),\\ (A^+J)&=\int\int dy_1dy_2\ A^+(y_1,y_2)J(y_1,y_2)=(AJ^+)\\ &=\int\int dy_1dy_2\ \Phi^+(y_1)A^+(y_1,y_2)\sqrt{D_m(y_1-y_2)}\Phi(y_2)\\ &=(\Phi^+(A^+\sqrt{D_m})\Phi).\end{aligned}$$

We can always represent the product $\Phi^+(x_1)\Phi(x_2)\Phi^+(x_3)\Phi(x_4)$ as a combination of currents

$$g^2\Phi^+(x_1)\Phi(x_2)\Phi^+(x_3)\Phi(x_4)\to g^2J(x_1,x_2)J(x_3,x_4).$$

We have

$$\begin{aligned}&G(x_1,x_2;x_3,x_4)\\ &=\int D\Phi D\Phi^+\int DA\ g^2J(x_1,x_2)J(x_3,x_4)\cdot e^{-(\Phi^+D_M^{-1}\Phi)-\frac{1}{2}(A^+A)-g\ (A^+J)}.\end{aligned}$$

Now we integrate by parts over A and A^+ and calculate the Gaussian integral over Φ and Φ^+:

$$G(x_1,x_2;x_3,x_4)=\int\int DA\ A(x_1,x_2)A(x_3,x_4)\ e^{S[A]}, \tag{33}$$

$$S[A]=-\frac{1}{2}(A^+A)-\mathrm{tr}\ln\left[1+g(A^+\sqrt{D_m})D_M\right].$$

Matrix operations are defined by formulas

$$\left(A^+\sqrt{D_m}\right)_{x_1,x_2}=A^+(x_1,x_2)\sqrt{D_m(x_1-x_2)},$$

$$\left[(A^+\sqrt{D_m})D_M\right]_{x_1,x_2}=\int dy\ A^+(x_1,y)\sqrt{D_m(x_1-y)}D_M(y-x_2).$$

We would like to stress that the representation (33) is completely equivalent to the initial representation (13). The Green function $G(x_1,x_2;x_3,x_4)$ can be considered as the Green function of the bilocal field $A(x_1,x_2)$. These fields are described by the nonlocal action $S[A]$.

B. One-loop representation.

The next problem is to give the standard particle interpretation to the action $S[A]$ in (33). For this aim this action should be represented in the form

$$S[A]\to-\frac{1}{2}(A^+R^{-1}A)+I_{int}[A],\qquad I_{int}[A]=O(A^3).$$

It means that we have to remove the term linear in A and extract the quadratic term out of $S[A]$. Let us introduce the displacement

$$A(y_1, y_2) \to A(y_1, y_2) + A_0(y_1, y_2)$$

We get

$$S[A] = -E_0 - \frac{1}{2}(A^+ A) - (A^+ A_0) - \mathrm{tr}\ln\left[1 + g\left(A^+\sqrt{D_m}\right)\mathcal{D}\right], \tag{34}$$

$$\mathcal{D} = D_M \cdot \frac{1}{1 + g(A_0^+\sqrt{D_m})D_M}$$

where the matrix multiplication is implied.

The constant term E_0 is the vacuum energy in the lowest approximation:

$$E_0 = -\frac{1}{2}(A_0^+ A_0) + \mathrm{tr}\ln\left[1 + g\left(A_0^+\sqrt{D_m}\right)D_M\right]$$

and will be omitted in subsequent calculations.

1. Linear term.

The term linear in A should be equal to zero

$$(A^+ A_0) + g\ \mathrm{tr}[(A^+\sqrt{D_m})\mathcal{D}] = 0,$$

or

$$A_0(x_1, x_2) + g\sqrt{D_m(x_1 - x_2)}\mathcal{D}(x_1, x_2) = 0$$

Introducing the function

$$A_0(x_1, x_2) = A_0(x_1 - x_2) = \frac{a(x_1 - x_2)}{g\sqrt{D_m(x_1 - x_2)}}$$

we get the equation

$$a(x_1 - x_2) = -g^2 D_m(x_1 - x_2)\mathcal{D}(x_1 - x_2),$$

where

$$\mathcal{D} = D_M \cdot \frac{1}{1 + aD_M}, \qquad \tilde{\mathcal{D}}(k^2) = \frac{1}{M^2 + k^2 + \tilde{a}(k^2)}.$$

Finally, we arrive at the equation of the Schwinger-Dyson type:

$$\tilde{a}(k^2) = -g^2 \int \frac{dp}{(2\pi)^4} \cdot \frac{1}{(m^2 + (k-p)^2)(M^2 + p^2 + \tilde{a}(p^2))}. \tag{35}$$

The integral in this equation contains the logarithmic ultraviolet divergence which can be removed by the renormalization of the mass M. It means that we should put

$$M^2 + \tilde{a}(k^2) = M_r^2 + \tilde{a}_r(k^2, M_r^2),$$

where M_r is the "physical" mass of the constituent particle Φ and

$$\tilde{a}_r(k^2, M_r^2) = \tilde{a}(k^2) - \tilde{a}(-M_r^2).$$

It is convinient to define the dimensionless function

$$w(k^2) = \frac{\tilde{a}^{(r)}(k^2, M_r^2)}{M_r^2}$$

which satisfies the equation

$$w(k^2) = \frac{g^2}{M_r^2} \int \frac{dp}{(2\pi)^4} \cdot \left[\frac{1}{(m^2 + (q-p)^2)(M_r^2 + p^2 + M_r^2 w(p^2))} \Big|_{q^2 = -M_r^2} \right. \\ \left. - \frac{1}{(m^2 + (k-p)^2)(M_r^2 + p^2 + M_r^2 w(p^2))} \right]. \tag{36}$$

We have obtained the functional equation of the type

$$w(k^2) = F[w, k^2].$$

Solution can be found by the fixed point method, i.e. we choose the initial "point" $w_0(k^2)$ and calculate

$$w_{n+1}(k^2) = F[w_n, k^2], \qquad \text{for} \qquad n = 0, 1, 2, ...$$

In the limit $n \to \infty$ we get

$$w_n(k^2) \to w(k^2).$$

The main problem is to choose the zeroth approximation $w_0(k^2)$. We proceed in the following way. The propagator $\tilde{\mathcal{D}}(k^2)$ after the mass renormalization should behave for $k^2 \to M_r^2$ like

$$\tilde{\mathcal{D}}(k^2) = \frac{1}{M_r^2 + k^2 + M_r^2 w(k^2)} \to \frac{Z}{M_r^2 + k^2}, \tag{37}$$
$$Z = \frac{1}{1+c}, \qquad c = M_r^2 \frac{d}{dk^2} w(k^2) \Big|_{k^2 = -M_r^2}$$

Let us define the zeroth approximation

$$\tilde{\mathcal{D}}_{(0)}(k^2) = \frac{Z}{M_r^2 + k^2}, \qquad Z = \frac{1}{1+c}$$

where the constant c is determined by the equation

$$c = M_r^2 \frac{d}{dk^2} w_{(0)}(k^2) \Big|_{k^2 = -M_r^2}$$
$$= - M_r^2 \frac{d}{dk^2} \left\{ g^2 \int \frac{dp}{(2\pi)^4} \cdot \frac{1}{(m^2 + (k-p)^2)} \cdot \frac{Z}{M_r^2 + p^2} \right\} \Big|_{k^2 = -M_r^2}$$
$$= Z \frac{g^2}{16\pi M_r^2} \cdot \frac{1}{\pi} \int_0^1 \frac{d\alpha \ \alpha(1-\alpha)}{\xi^2 \alpha + (1-\alpha)^2} = \lambda Z \cdot a(\xi).$$

Here

$$a(\xi) = \frac{1}{\pi} \left[-1 - (1 - \xi^2) \ln \xi + \frac{\xi(3 - \xi^2)}{\sqrt{4 - \xi^2}} \cdot \arctan \left(\frac{\sqrt{4 - \xi^2}}{\xi} \right) \right]. \tag{38}$$

The equation

$$Z = \frac{1}{1+c} = \frac{1}{1 + \lambda Z a(\xi)}$$

gives

$$Z = Z_{(0)} = \frac{2}{1+\sqrt{1+4\lambda a(\xi)}}. \tag{39}$$

Thus, one can choose in the zeroth approximation

$$w_{(0)}(p^2) = \left(1+\frac{k^2}{M_r^2}\right)\left(\frac{1}{Z}-1\right).$$

In the first approximation (n=1) we have

$$w_{(1)}(k^2) = \lambda Z \int\limits_0^1 \frac{d\alpha}{\pi}\ \ln\left\{\frac{m^2\alpha + M_r^2(1-\alpha)+\alpha(1-\alpha)k^2}{m^2\alpha+M_r^2(1-\alpha)^2}\right\}$$

where

$$c_{(1)} = M_r^2 \frac{d}{dk^2} w_{(1)}(k^2)\Big|_{k^2=-M_r^2} = \lambda Z_{(0)} a(\xi) = c,$$
$$Z_{(1)} = \frac{1}{1+c_{(1)}} = Z_{(0)}.$$

Finally, we have in the zeroth and first approximations

$$\tilde{\mathcal{D}}_{(0)}(k^2) = \frac{Z}{M_r^2+k^2}, \tag{40}$$
$$\tilde{\mathcal{D}}_{(1)}(k^2) = \frac{1}{M_r^2+k^2+M_r^2 w_{(1)}(k^2)} \to \frac{Z}{M_r^2+k^2} \quad \text{for} \quad k^2 \to M_r^2.$$

One can check that

$$\tilde{\mathcal{D}}_{(0)}(k^2) \approx \tilde{\mathcal{D}}_{(1)}(k^2)$$

with 5% accuracy.

The renormalized coupling constant g_r can be defined by a standard way:

$$g_r = gZ, \qquad \lambda_r = \lambda Z^2 \tag{41}$$

because

$$g\tilde{D}(k^2) = \frac{g}{M_r^2+k^2+M_r^2 w(k^2)} \to \frac{gZ}{M_r^2+k^2} \quad \text{for} \quad k^2 \to M_r^2.$$

In subsequent numerical calculations we use the zeroth approximation $D_{(0)}$ (40) which gives quite acceptable qualitative semiquantative estimations.

In this approximation the renormalized coupling constant is

$$\lambda_r = \lambda_r(\lambda,\xi) = \lambda Z^2 = \frac{4\lambda}{(1+\sqrt{1+4\lambda a(\xi)})^2} \tag{42}$$

It is important that the renormalization considerably diminishes the initial coupling constant λ. For example, in the case of the "deuteron", when Φ-particle is the "proton", ϕ-paricle is the "π-meson" and $\lambda = 14.5$

$$\lambda_r = \lambda_r\left(14.5, \frac{m_\pi}{M_p}\right) = 1.712.$$

Moreover, the renormalized coupling constant λ_r is bounded for any λ

$$\lambda_r \le \frac{1}{a(\xi)} \le \frac{1}{a(1)} = 15.01....$$

After removing the linear term we have

$$S[A] = -\frac{1}{2}(A^+A) - \mathrm{tr}\ln_1\left[1 + g(A^+\sqrt{D_m})\mathcal{D}\right],$$
$$\ln_1(1+Q) = \ln(1+Q) - Q.$$

The "trace" of "ln" consists of terms

$$\int\int dx_j dx_{j+1}\ \mathcal{D}(... - x_j)A^+(x_j, x_{j+1})\sqrt{D_m(x_j - x_{j+1})}\mathcal{D}(x_{j+1} - ...) \tag{43}$$

Let us proceed as follows. We introduce new variables

$$x_j = z_j + \frac{y_j}{2}, \qquad x_{j+1} = z_j - \frac{y_j}{2};$$

and the notation

$$A(x_j, x_{j+1}) = W(z_j, y_j), \quad A(x_{j+1}, x_j) = A^+(x_j, x_{j+1}) = W(z_j, -y_j).$$

The term (43) can be written as

$$\int\int dz_j dy_j\ D\left(... - z_j - \frac{y_j}{2}\right)W(z_j, -y_j)\sqrt{D_m(y_j)}D\left(z_j - \frac{y_j}{2}...\right) \tag{44}$$
$$= \int dz_j\ D(... - z_j)\mathcal{V}(z_j, \overleftrightarrow{p}_{z_j})D(z_j - ...)$$

where

$$\mathcal{V}(z, \overleftrightarrow{p}_z) = \int dy\ \sqrt{D_m(y)}W(z, -y)e^{-i\frac{y}{2}\overleftrightarrow{p}_z},$$

$$\overleftrightarrow{p}_x = \frac{1}{i}\left[\overleftarrow{\partial}_x - \overrightarrow{\partial}_x\right]$$

Let the system of functions $\{U_Q(y)\}$ with quantum numbers $Q = (nl\{\mu\})$, where n, l and $\{\mu\}$ are radial, orbital and magnetic quantum numbers, be orthonormal, i.e.,

$$(U_Q U^*_{Q'}) = \int d^4y\ U_Q(y)U^*_{Q'}(y) = \delta_{QQ'} = \delta_{nn'}\delta_{ll'}\delta_{\{\mu\}\{\mu'\}}, \tag{45}$$
$$\sum_Q U_Q(y)U^*_Q(y') = \delta(y - y'),$$
$$U_Q(-y) = U^*_Q(y).$$

The function W can be represented by

$$W(z, -y) = \sum_Q W_Q(z)U_Q(-y), \qquad W^*_Q(z) = W_Q(z), \tag{46}$$
$$\tilde{W}(p, -y) = \sum_Q \tilde{W}_Q(p)U_Q(-y), \qquad \tilde{W}^*_Q(p) = \tilde{W}_Q(p),$$
$$W_Q(z) = \int \frac{dp}{(2\pi)^4}\cdot\tilde{W}_Q(p)\ e^{-ipz}.$$

Then we have

$$\mathcal{V}(z,\overleftrightarrow{p}_z) = \sum_Q W_Q(z)\int dy\ \sqrt{D_m(y)}U_Q(y)e^{i\frac{y}{2}\overleftrightarrow{p}_z} \tag{47}$$
$$= \sum_Q W_Q(z)V_Q\left(\overleftrightarrow{p}_x\right) = (WV)_z,$$
$$V_Q\left(\overleftrightarrow{p}_x\right) = \int dy\sqrt{D_m(y)}U_Q(y)e^{-i\frac{y}{2}\overleftrightarrow{p}_x}.$$

In this notation one obtains

$$(A^+A) = \int dx_1\int dx_2\ A^+(x_1,x_2)A(x_2,x_1) = (WW)$$
$$= \sum_Q\int dz\ W_Q(z)W_Q(z) = \sum_Q\int dp\ \tilde{W}_Q(p)\tilde{W}_Q(p),$$

$$\mathrm{tr}\ln_1\left[1+g(A^+\sqrt{D_m})\mathcal{D}\right] = \mathrm{tr}\ln_1(1+g_r(WV)\mathcal{D}_r).$$

The basic representation for the Green functions under consideration becomes of the form

$$G(x_1,x_2;x_3,x_4)\to G_{Q_1Q_2}(z_1,z_2) = \int\prod_Q DW_Q\cdot W_{Q_1}(z_1)W_{Q_2}(z_2)\cdot e^{\mathcal{S}[W]}, \tag{48}$$
$$\mathcal{S}[W] = -\frac{1}{2}(WW) - \mathrm{tr}\ln_1\left[1+g_r(WV)\mathcal{D}_r\right].$$

C. Particle interpretation of the quadratic term.

Let us extract the quadratic form from $\mathcal{S}[W]$

$$\mathcal{S}[W] = -\frac{1}{2}(W[I-g_r^2\Pi]W) - \mathrm{tr}\ln_2[1+g_r(WV)\mathcal{D}_r], \tag{49}$$
$$\ln_2(1+s) = \ln(1+s) - s + \frac{s^2}{2}.$$

Here

$$g_r^2\Pi = g_r^2\mathrm{tr}[V\mathcal{D}_rV\mathcal{D}_r] \tag{50}$$

and according to (46) and (47)

$$(Wg_r^2\Pi W) = \sum_{QQ'}\iint dxdx'\ W_Q(x)g_r^2\Pi_{QQ'}(x-x')W_Q(x')$$
$$= \sum_Q\int dp\ \tilde{W}_Q^*(p)g_r^2\tilde{\Pi}_{QQ'}(p)\tilde{W}_Q(p).$$

The polarization operator $g_r^2\tilde{\Pi}_{QQ'}$ looks

$$g_r^2\Pi_{QQ'}(x-x') = g_r^2\mathrm{tr}\left\{V_Q\mathcal{D}_rV_{Q'}\mathcal{D}_r\right\} \tag{51}$$
$$= g_r^2V_Q\left(\overleftrightarrow{p}_x\right)\mathcal{D}_r(x-x')V_{Q'}\left(\overleftrightarrow{p}_{x'}\right)\mathcal{D}_r(x'-x),$$
$$= g_r^2\iint dydy'\ U_Q(y)P(x-x';y,y')U_{Q'}^*(y'),$$
$$P(x;y,y') = \sqrt{D_m(y)}\mathcal{D}_r\left(x-\frac{y-y'}{2}\right)\mathcal{D}_r\left(x+\frac{y-y'}{2}\right)\sqrt{D_m(y')},$$
$$\tilde{P}(p;y,y') = \int dx\ e^{ipx}P(x;y,y').$$

In the momentum space we get

$$g_r^2\tilde{\Pi}_{QQ'}(p) = g_r^2 \int \frac{dk}{(2\pi)^4}\, V_Q(k)\tilde{\mathcal{D}}_r\left(k+\frac{p}{2}\right)\tilde{\mathcal{D}}_r\left(k-\frac{p}{2}\right)V_{Q'}(k). \tag{52}$$

In our approximation we have

$$g_r^2\tilde{\Pi}_{QQ'}(p) = g_r^2 \int \frac{dk}{(2\pi)^4}\cdot\frac{V_Q(k)V_{Q'}(k)}{\left(M_r^2+\left(k+\frac{p}{2}\right)^2\right)\left(M_r^2+\left(k-\frac{p}{2}\right)^2\right)}. \tag{53}$$

The orthonormal system $\{U_Q(x)\}$ should be chosen so that the polarization operator $\tilde{\Pi}_{QQ'}(p)$ should be diagonal in radial (n,n') and orbital (l,l') quantum numbers

$$\tilde{\Pi}_{QQ'}(p) = \delta_{nn'}\delta_{ll'}\cdot\tilde{\Pi}^{(nl)}_{\{\mu\}\{\mu'\}}(p) \tag{54}$$

The index structure of the diagonal polarization operator $\tilde{\Pi}^{(nl)}_{\{\mu\}\{\mu'\}}(p)$ looks like

$$\tilde{\Pi}^{(nl)}_{\{\mu\}\{\mu'\}}(p) = \tilde{\Pi}^{(nl)}_{\{\mu\}\{\mu'\}}(p) = \tilde{\Pi}^{(nl)}(p^2)\cdot\delta_{\{\mu\}\{\mu'\}} + \sum_j \tilde{\Pi}^{(nl)}_j(p^2)\cdot t^j_{\{\mu\}\{\mu'\}}(p)$$

where the tensors $t^j_{\{\mu\}\{\mu'\}}(p)$ contain combinations of the vectors $p_\mu p_{\mu'}$.

The diagonal quadratic form of (49) gives the equation of motion for the field $W_Q(x) = W^{(nl)}_{\{\nu\mu_2\ldots\mu_l\}}(x)$

$$\left[\delta_{QQ'} - g_r^2\tilde{\Pi}_{QQ'}\left(\frac{\partial}{i\partial x}\right)\right]W_{Q'}(x) = 0,$$

or

$$\left[\delta_{QQ'} - g_r^2\tilde{\Pi}_{QQ'}(p)\right]\tilde{W}_{Q'}(p) = 0. \tag{55}$$

The requirement that this equation on the mass shell should be the Klein-Gordon equation gives the constraint

$$\frac{\partial}{\partial x_\nu}W^{(nl)}_{\nu\mu_2\ldots\mu_l}(x) = 0 \qquad \text{or} \qquad p_\nu\tilde{W}^{(nl)}_{\{\nu\mu_2\ldots\mu_l\}}(p) = 0$$

on the mass shell. Thus, the function $\tilde{W}^{(nl)}_{\{\mu\}}(p)$ satisfies the equation

$$\left[1 - g_r^2\tilde{\Pi}^{(nl)}(p^2)\right]\tilde{W}^{(nl)}_{\{\mu_1\ldots\mu_l\}}(p) = 0. \tag{56}$$

The mass of the state with quantum numbers $Q = (nl)$ is defined by the equation

$$1 - g_r^2\tilde{\Pi}^{(nl)}(-M^2_{(nl)}) = 0. \tag{57}$$

Let us write

$$-1 + g_r^2\tilde{\Pi}^{(nl)}(p^2) = -Z_{(nl)}(p^2 + M^2_{(nl)}) + \Sigma^{(nl)}(p^2),$$

$$\Sigma^{(nl)}(p^2) = g_r^2\tilde{\Pi}^{(nl)}_{reg}(p^2, M^2_{(nl)})$$
$$= g_r^2\left[\tilde{\Pi}^{(nl)}(p^2) - \tilde{\Pi}^{(nl)}(-M^2_{(nl)}) - \tilde{\Pi}'_{(nl)}(-M^2_{(nl)})(p^2 + M^2_{(nl)})\right],$$

$$Z_{(nl)} = g_r^2\left[-\tilde{\Pi}'_{(nl)}(-M^2_{(nl)})\right]$$

The constant $Z_{(nl)}$ is positive.

New field variables can be introduced as follows:

$$W_Q(x) = \frac{\varphi_Q(x)}{\sqrt{Z_{(nl)}}}. \tag{58}$$

The representation (48) assumes of the form

$$G_{Q_0}(x-y) = \frac{1}{Z_{Q_0}}\mathcal{G}_{Q_0}(x-y), \tag{59}$$
$$\mathcal{G}_{Q_0}(x-y) = \int D\varphi\ \varphi_{Q_0}(x)\varphi_{Q_0}(y)\ e^{\mathbf{S}[\phi]},$$

where the action $\mathbf{S}$ looks like

$$\mathbf{S}[\phi] = -\frac{1}{2}(\varphi\mathcal{D}^{-1}\varphi) - \mathcal{I}_{int}[\varphi] \tag{60}$$

and the appropriate normalization should be chosen.

The kinetic term is

$$\begin{aligned}(\varphi\mathcal{D}^{-1}\varphi) &= \left(\varphi\left[-\Box + M_b^2 + \Sigma_b\right]\varphi\right) \\ &= \int dx \sum_{\mathcal{Q}} \varphi_Q(x)\left[-\Box + M_{(nl)}^2 + \Sigma^{(nl)}(-\Box)\right]\varphi_Q(x) \\ &= \int dp \sum_{\mathcal{Q}} \tilde{\varphi}_Q^+(p)\left[p^2 + M_{(nl)}^2 + \Sigma^{(nl)}(p^2)\right]\tilde{\varphi}_Q(p)\end{aligned} \tag{61}$$

and the interaction term is

$$\begin{aligned}&\mathcal{I}_{int}[\varphi] = \mathrm{tr}\ln_2\left[1 + (h\varphi V)\mathcal{D}\right], \\ &(h\varphi V) = \sum_Q h_Q\varphi_Q V_Q, \qquad h_Q = \frac{1}{\sqrt{-\Pi'_Q(-M_Q^2)}}\end{aligned} \tag{62}$$

The effective dimensionless coupling constants are defined as

$$\lambda_Q^{(eff)} = \frac{h_Q^2}{16\pi M_r^2} = \frac{1}{16\pi[-M_r^2\tilde{\Pi}'_{(nl)}(-M_{(nl)}^2)]}. \tag{63}$$

As a result, the final representation (59) can be interpreted as a partition function of the quantum field system of bosonic fields $\{\phi_Q\}$ which have masses M_Q and are described by the nonlocal action (60).

We would like to stress that the resulting representation for the generating functional does not contain the initial coupling constant g.

All calculations with the generating functional (59) can be performed by perturbation expansions in coupling constsnts h_Q. We can trust these calculations if and only if the effective coupling constants (63) are small enough:

$$\lambda_Q^{(eff)} \ll 1.$$

D. The orthonormal system.

The next step is to determine the orthonormal system (45). The problem is to find the spectrum and eigenfunctions of the operator $\tilde{P}(p; y, y')$ in (51), i.e.

$$\tilde{P}(p)U_Q = E_Q(p^2)U_Q$$

or

$$\int dy'\tilde{\mathcal{P}}(p; y, y')U_Q(y', p) = E_Q(p)U_Q(y, p), \qquad Q = (n, l, \{\mu\}). \tag{64}$$

This equation can be represented in a standard form of the Bethe-Salpeter equation in the one-boson exchange approximation. Using the relation

$$K_+K_- \cdot \int dx\ e^{ipx}\mathcal{D}_r\left(x-\frac{y-y'}{2}\right)\mathcal{D}_r\left(x+\frac{y-y'}{2}\right)=\delta(y-y')$$

with

$$K_\pm=\left[M_r^2+\left(i\frac{\partial}{\partial y}\pm\frac{p}{2}\right)^2\right]$$

and introducing the functions

$$\Psi_Q(y,p)=\frac{1}{\sqrt{D_m(y)}}\cdot U_Q(y,p)$$

we get the standard form of the Bethe-Salpeter equation (see, for example, [1])

$$\left[M_r^2+\left(i\frac{\partial}{\partial y}+\frac{p}{2}\right)^2\right]\cdot\left[M_r^2+\left(i\frac{\partial}{\partial y}-\frac{p}{2}\right)^2\right]\Psi_Q(y,p)=g_r^2D_m(y)\Psi_Q(y,p), \tag{65}$$

where the spectrum is defined by the equation

$$g_r^2E_Q(-M_Q^2)=1. \tag{66}$$

Thus the diagonalization of the operator $\tilde{P}(p;y,y')$ is equivalent to the solution of the Bethe-Salpeter equation in one-boson exchange approximation.

It should be noted that in this formulation the functions $\Psi_Q(y,p)$ depend on the direction of the vector $p=(p_0,\vec{p})$ in the space $\mathbf{R}^4$, the standard choice is $p_{nl}=(iM_{nl},0)$.

The main problem to use the Bethe-Salpeter basis is that the Bethe-Salpeter equation can only be solved by numerical methods. Even the solution obtained by Wick and Cutkosky [1,7] is reduced to the differential equation which should be numerically computed. Our aim is to continue analytic calculations as long as possible in order to get a visible general picture of arising bound states in the system under consideration. Therefore we choose a more practical way, namely, we use an orthonormal basis that is simple enough from an analytic point of view and is directly connected with the problem under consideration. In this case the operators $g_r^2\tilde{\Pi}_{QQ'}$ are not diagonal so that we should diagonalize them. The idea consists in finding an effective basis for diagonalization of $g_r^2\tilde{\Pi}_{QQ'}$ such that its lowest function would provide a good qualitative description for the eigenvelues $E_{(nl)}$ and the next two or three functions only give a good quantitive description for those eigenvelues.

This effective basis $\{U_Q(x)\}$ can be constructed using the boson Green function $D_a(u)$ (5) with a parameter a as a weight function inducing uniquely the system of orthonormal polynomials in the space R^4. Thus, the full orthonormal system of functions (45) can be chosen in the form

$$U_Q(x,a)=i^l\sqrt{D_a(x)}aP_Q(ax). \tag{67}$$

Here $P_Q(u)$ are real polynomials, satisfying

$$P_Q(-u)=(-1)^lP_Q(u),\quad \text{so that}\quad U_Q^*(x)=U_Q(-x).$$

The parameter a that enters into the orthonormal system is fixed by a variation condition formulated below.

The construction of this basis is presented in the Appendix A.

E. The diagonalization procedure.

Let us demonstrate our procedure of diagonalization of the polarization operators (53) for states $Q=(n0)$ and $Q'=(n'0)$. In the momentum space this polarization operator looks like

$$g_r^2\tilde{\Pi}_{(nn')}(p^2)=g_r^2\int\frac{dk}{(2\pi)^4}\cdot\frac{V_{(n0)}(k|m,a)V_{(n'0)}(k|m,a)}{\left(M_r^2+\left(k+\frac{p}{2}\right)^2\right)\left(M_r^2+\left(k-\frac{p}{2}\right)^2\right)} \tag{68}$$

where the vertex defined by (47) looks in this case like

$$V_{(n0)}(p|m,a) = \int dx\ U_{(n0)}(x)\sqrt{D_m(x)}e^{ixp} \tag{69}$$
$$= \int dx\ \sqrt{D_m(x)D_a(x)}aP_{(n0)}(ax)e^{ixp}.$$

One can develop

$$\sqrt{D_{m_1}(x)D_{m_2}(x)} = \sum_{n=0}^{\infty}\Delta^{2n}C_n\left(\frac{\Delta}{m}\right)D_m^{(n)}(x) = D_m(x)\left[1+O\left(\frac{\Delta^2}{m^2}\right)\right]$$

with $m = \frac{m_1+m_2}{2}$ and $\Delta = \frac{m_1-m_2}{2}$ and

$$D_m^{(n)} = \int \frac{dk}{(2\pi)^4}\cdot\frac{e^{ikx}}{(m^2+k^2)^{1+n}}.$$

We shall use the lowest approximation

$$\sqrt{D_m(x)D_a(x)} \approx D_{\frac{m+a}{2}}(x), \tag{70}$$

the accuracy of which is quite acceptable for our consideration. Then, the vertex acquires the form

$$V_{(n0)}(p^2|m,a) = P_{(n0)}\left(-a^2\Box_p\right)\cdot\frac{\mu}{\left(\frac{m+a}{2}\right)^2+p^2}. \tag{71}$$

The explicit form of the polynomials $P_{(n0)}(u^2)$ is given in the Appendix A. We get

$$g_r^2\tilde{\Pi}_{(nn')}(p^2)$$
$$= \frac{\lambda_r}{2\pi}\cdot\int_0^1 dt\ R_{(n)}(t)R_{(n')}(t)\cdot\frac{1}{b}\left[\sqrt{1+\frac{4bt(1-t)}{(1-bt)^2}}-1\right],$$
$$R_{(n)}(t) = \frac{M_r}{t}V_{(n0)}\left(M_r^2\frac{1-t}{t}\middle| m,a\right).$$

Now we formulate the variational principle which defines the parameter a. The operator $\tilde{\Pi}_{(00)}$ is the largest eigenvalue of the operator matrix $\tilde{\Pi}_{(nn')}$, therefore the parameter a can be defined by the variation requirement

$$\max_a\ g_r^2\ \tilde{\Pi}_{(00)}(-M_{(0)}^2) \tag{72}$$
$$= \max_\eta\ \frac{\lambda_r}{2\pi}\int_0^1 dt\ \left(\frac{\eta}{\left(\frac{\xi+\eta}{2}\right)^2 t+1-t}\right)^2\cdot\frac{1}{b}\left[\sqrt{1+\frac{4bt(1-t)}{(1-bt)^2}}-1\right]$$

where the notion

$$b = \left(\frac{M_{(0)}}{2M_r}\right)^2, \qquad \xi = \frac{m}{M_r}, \qquad \eta = \frac{a}{M_r}$$

are used. Thus, the parameter $a = a(M_{(0)}, m)$ is a function of m and $M_{(0)}$. Quite a good approximation is as follows

$$\eta = \eta(\xi, b) \approx 2 - \frac{3}{2}b + \frac{3}{2}\xi. \tag{73}$$

Let us show that this orthonormal functions with the parameter a (73) gives quite good approximation for the eigenvalues of the matrix $\tilde{\Pi}_{(nn')}(p^2)$. For this aim we calculate the matrix

$$\mathcal{P}_N = \left\{\tilde{\Pi}_{(nn')}(b,\xi,\eta), \quad (n,n'=0,...,N)\right\}$$

and their eigenvalues

$$\mathcal{E}_N = \mathrm{diag}\left\{E_0^{(N)}, ..., E_N^{(N)}\right\}.$$

Then we have to compare $E_j^{(N)}$ for fixed j and different N.

The numerical results are given in Table I. The first case for $\xi = .5,\ b = .25,\ \eta = 2.451$ and the second case for $\xi = .2,\ b = .9,\ \eta = 1.22$. One can see that for the lowest eigenvalue practically the first lowest eigenfunction can be used, i.e. our choice of the orthonormal system gives quite a good accuracy.

Table I. Diagonalization of the matrix $\mathcal{P}_N$.

N	E_0	E_1	E_2	E_3
0	.04165			
1	.04166	.009941		
2	.04173	.010279	.002755	
3	.04175	.010368	.003295	.0007482
4	.04175	.010402	.003546	.0010336
0	.1239			
1	.1262	.03564		
2	.1262	.03616	.01162	
3	.1263	.03645	.01298	.003789
4	.1263	.03655	.01373	.004710

F. Applicability of S-, BS- and BNC- methods.

Three parameters ξ, λ_r and b are not independent. The standard formulation of the problem is the following: the parameters ξ and λ_r are given and the mass of a bound state b has to be found. We reformulate this problem: what is the region of changing b for the fixed parameter ξ, if the effective coupling constant λ_{eff} is smaller then one?

In this section we give qualitative curves

$$\frac{\Delta M}{2M} = 1 - \frac{M_{(0)}(\xi)}{2M_r} = 1 - \sqrt{1 - b(xi)}$$

which restrict the admissible regions of ξ and b when the effective coupling constant in S-, BS- and BNC-methods is smaller then 1. These curves are presented in Fig 1. and show the applicability of S-, BS- and BNC-methods to study the bound state problem.

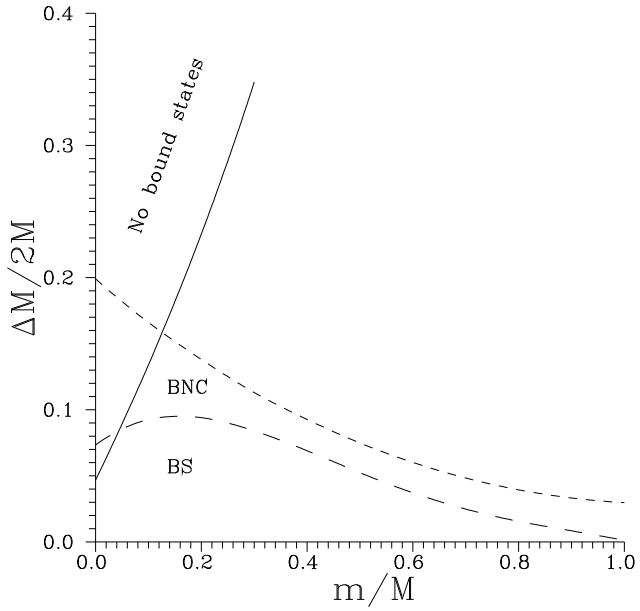

FIG. 1. Applicability of S-, BS- and BNC- methods.

- The region of applicability of the nonrelativistic Schrödinger equation is a small vicinity near point $(0,0)$ in Fig 1.

- The range of existence of bound states (under solid line in Fig 1) is defined by the equation

$$1 = \lambda_{max}(\xi)\Pi_{(0)}(\xi, b)$$

where

$$\lambda_{max}(\xi) = \max_{\lambda} \lambda_r(\lambda, \xi) = \frac{1}{a(\xi)},$$

$$\Pi_{(0)}(\xi, b)$$

$$= \max_{\eta} \int_0^1 \frac{dt}{2\pi} \left(\frac{\eta}{\left(\frac{\xi+\eta}{2}\right)^2 t + 1 - t} \right)^2 \cdot \frac{1}{b} \left[\sqrt{1 + \frac{4bt(1-t)}{(1-bt)^2}} - 1 \right].$$

- The region of applicability of the Bethe-Salpeter method (under dashed line in Fig 1) is defined by $\lambda < 1$ and

$$\lambda_r \leq \lambda_1(\xi) = \lambda_r(1, \xi) = \frac{4}{(1 + \sqrt{1 + 4a(\xi)})^2}$$

where $b = b(\xi)$ is defined by the equation

$$1 = \lambda_1(\xi)\Pi_{(0)}(\xi, b(\xi)).$$

- The rigion of applicability of the BNC-method (under dotted line in Fig 1) is defined by the inequality

$$\lambda_0^{(eff)}(\xi, b) \leq 1$$

where the effective coupling constant is defined by (63):

$$\lambda_0^{(eff)} = \frac{1}{16\pi[-M_r^2 \tilde{\Pi}'_{(0)}(-M_{(0)}^2)]} = \frac{1}{\Phi(\xi, b)},$$

$$\Phi(\xi,b)=\int\limits_0^1\frac{dt}{2\pi}\left(\frac{\eta(\xi,b)}{\left(\frac{\xi+\eta(\xi,b)}{2}\right)^2t+1-t}\right)^2$$
$$\cdot\ \frac{1}{b^2}\left\{1-\frac{1}{\sqrt{1+\frac{4bt(1-t)}{(1-bt)^2}}}\cdot\left[1+\frac{2bt(1-t)(1-3bt)}{(1-bt)^3}\right]\right\}.$$

G. Small binding energies.

In this section we show that the BNC-method gives the correct nonrelativistic limit in the case of the Coulomb potential. For the Yukawa potential we show how the screening length depends on the coupling constant and the patameter $\xi=\frac{m}{M}$. If the binding energy is small then

$$\epsilon=1-b=1-\left(\frac{M_b}{2M}\right)^2\approx\frac{2M-M_b}{M}=\frac{\Delta M}{M}\ll 1.$$

and one can get for (72) when ϵ is small

$$\frac{1}{2\pi}\int\limits_0^1 dt\left(\frac{\eta}{\left(\frac{\xi+\eta}{2}\right)^2t+1-t}\right)^2\cdot\frac{1}{b}\left[\sqrt{1+\frac{4bt(1-t)}{(1-bt)^2}}-1\right]$$
$$=\frac{\epsilon}{2\pi}\int\limits_0^{1/\epsilon}du\left(\frac{\eta}{\left(\frac{\xi+\eta}{2}\right)^2(1-\epsilon u)+\epsilon u}\right)^2\cdot\left[\sqrt{1+\frac{4u(1-\epsilon u)}{\epsilon(1+u)^2}}-1\right]$$
$$\to\frac{1}{\epsilon^{3/2}\pi}\int\limits_0^\infty du\sqrt{u}\left(\frac{\eta}{\left(\frac{\xi+\eta}{2\sqrt{\epsilon}}\right)^2+u}\right)^2\cdot\frac{1}{1+u}=\frac{4\eta^2}{(\xi+\eta)(\xi+\eta+2\sqrt{\epsilon})^2}.$$

The Coulomb potential $H=\frac{p^2}{2M_c}-\frac{\alpha}{r}$ in our approach is defined by the conditions $m=0$ or $\xi=0$ and $\alpha\ll 1$. We get

$$\Pi_{(0)}(0,\epsilon)=\max_\eta\frac{4\eta}{(\eta+2\sqrt{\epsilon})^2}=\frac{1}{2\sqrt{\epsilon}}$$

and from

$$1=\lambda_r\Pi_{(0)}(0,\epsilon),\quad\text{or}\quad 1=\frac{\alpha}{2\sqrt{\epsilon}},\qquad(\lambda_r=\alpha)$$

it follows that

$$\Delta M=\frac{\alpha^2}{2}\frac{M_r}{2}=\frac{\alpha^2}{2}M_{cm}$$

according with the lowest energy of the nonrelativistic Schrödinger equation for the Coulomb potential.

For the Yukawa potential $H=\frac{p^2}{2M_c}-\frac{\alpha}{r}e^{-mr}$ the critical screening length, defined by the condition $\epsilon=0$ for the lowest state, can be computed. For small ξ this length is defined by the equation

$$1=\alpha\max_\eta\frac{4\eta^2}{(\xi_c+\eta)^3}=\frac{16}{27}\cdot\frac{\alpha}{\xi_c}$$

and

$$m_c=\frac{32}{27}\alpha\frac{M_r}{2}=1.1851...\cdot\alpha M_{cm}.$$

or

$$\kappa_c = \frac{m_c}{\alpha M_{cm}} = \frac{32}{27} = 1.1851...$$

The numerical calculations [18] give

$$m_c = 1.1906... \cdot \alpha M_{cm}.$$

In the case $0 < \xi \leq 1$ the relation between $\xi_c = \frac{m_c}{M_r}$ and α_c is given by

$$\alpha_c \cdot T(\xi_c) = 1 \qquad \text{and} \qquad \kappa_c = 2\xi_c T(\xi_c)$$

where

$$T(\xi_c) = \max_{\eta} \frac{1}{2\pi} \int_0^1 dt \ \left(\frac{\eta}{\left(\frac{\xi_c+\eta}{2}\right)^2 t + 1 - t} \right)^2 \cdot \left[\sqrt{1 + \frac{4t}{1-t}} - 1 \right].$$

The results are given in Table II. One can see that for $\xi_c > .1$ the relativistic corrections are more then 5%.

Table II. The critical screening length for the Yukawa potential.

$\xi_c = \frac{m_c}{M_r}$	κ_c	$\alpha_c = \frac{1}{T(\xi_c)}$
0.0001	1.1851	0.00016875
0.001	1.1846	0.0016888
0.01	1.1795	0.01695
0.1	1.1324	0.1766
0.2	1.0861	0.368
0.4	1.0079	0.793
0.6	0.9436	1.3735
0.8	0.8893	1.799
1.0	0.8426	2.373

V. APPENDIX A. ORTHONORMAL SYSTEM

The orthonormal system $\{U_Q(x)\}$ is defined by

$$U_Q(x) = i^l \sqrt{D_a(x)} a P_Q(ax),$$
$$D_a(x) = \frac{a^2}{(2\pi)^2} \cdot \frac{1}{ax} K_1(ax), \qquad x = \sqrt{x^2}.$$

$$Q = (nl\{\mu\}), \qquad \{\mu\} = \mu_1, ..., \mu_l.$$

$$\int d^4x \ U_Q^*(x) U_{Q'}(x) = \int d^4x \ D_a(x) \ aP_Q(ax) \ aP_{Q'}(ax) = \delta_{QQ'},$$

$$\int d^4x = \int_0^\infty dr r^3 \int dn, \qquad \int dn = \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin\theta \int_0^\pi d\gamma \sin^2\gamma$$

$$\delta_{QQ'} = \delta_{nn'} \delta_{ll'} \delta_{\{\mu\}\{\mu'\}}, \qquad \sum_{\{\mu'\}} \delta_{\{\mu\}\{\mu'\}} F_{\{\mu'\}} = F_{\{\mu\}}.$$

The polynomials are defined by

$$aP_Q(ax) = aN_{(nl)}P_{(nl)}(a^2x^2)T^{(l)}_{\{\mu\}}(ax) = aN_{(nl)}P_{(nl)}(u^2)(\sqrt{u^2})^l T^{(l)}_{\{\mu\}}(n)$$

$$u = ax, \qquad n = \frac{x}{\sqrt{x^2}} = \frac{u}{\sqrt{u^2}}.$$

The orthonormality condition looks as

$$\begin{aligned}
&\int dx\ D_a(x)aP_Q(ax)aP_{Q'}(ax) \\
&= \frac{a^4}{(2\pi)^2}\int\limits_0^\infty dx x^3\ \cdot\frac{1}{ax}K_1(ax)\int dn\ P_Q(ax)P_{Q'}(ax) \\
&= \frac{1}{(2\pi)^2}\int\limits_0^\infty du u^2\ K_1(u)\int dn\ P_Q(u)P_{Q'}(u) \\
&= \frac{N^2_{(nl)}}{(2\pi)^2}\cdot 2\delta_{nn'}\cdot\frac{2\pi^2}{2^l(l+1)}\delta_{ll'}\cdot\delta_{\{\mu\}\{\mu'\}} = \delta_{QQ'}, \qquad N_{(nl)} = 2^{\frac{l}{2}}\sqrt{l+1},
\end{aligned}$$

$$\begin{aligned}
&\int dn\ T^{(l)}_{\{\mu\}}(n)T^{(l')}_{\{\mu'\}}(n) = \frac{2\pi^2}{2^l(l+1)}\cdot\delta_{ll'}\delta_{\{\mu\}\{\mu'\}}, \\
&\frac{1}{2}\int\limits_0^\infty ds\ s^{2+2l}K_1(s)P_{(nl)}(s^2)P_{(n'l)}(s^2) = \delta_{nn'}
\end{aligned}$$

A. Angular polynomials $T^{(l)}_{\{\mu\}}(n)$

Polynomials $T^{(l)}_{\{\mu\}}(n)$ satisfy

- $T^{(l)}_{\mu_1,\ldots,\mu_l}(n)$ is symmetric for $\mu_i \rightleftharpoons \mu_j$,
- $T^{(l)}_{\mu,\mu,\mu_3,\ldots,\mu_l}(n) = 0$,
- $T^{(l)}_{\mu_1\ldots\mu_l}(n) = \frac{1}{l}P(1|2\ldots l)T^{(l-1)}_{\mu_2\ldots\mu_l}(n) - \frac{1}{2l(l-1)}P(12|3\ldots l)\delta_{\mu_1\mu_2}T^{(l-2)}_{\mu_3\ldots\mu_l}(n)$.

The first four polynomials are

$$\begin{aligned}
&T^{(0)} = 1, \qquad T^{(1)}_\mu(n) = n_\mu, \\
&T^{(2)}_{12}(n) = n_1n_2 - \frac{1}{4}\delta_{12}, \\
&T^{(3)}_{123}(n) = n_1n_2n_3 - \frac{1}{6}(n_1\delta_{23} + n_2\delta_{31} + n_3\delta_{12}), \\
&T^{(4)}_{1234}(n) = n_1n_2n_3n_4 \\
&-\frac{1}{8}(n_1n_2\delta_{34} + n_1n_3\delta_{24} + n_1n_4\delta_{23} + n_2n_3\delta_{14} + n_2n_4\underset{\cdot}{e}lta_{13} + n_3n_4\delta_{12}) \\
&+\frac{1}{48}(\delta_{12}\delta_{34} + \delta_{13}\delta_{24} + \delta_{14}\delta_{23})
\end{aligned}$$

where the condensed notation $n_j = n_{\mu_j}$ and $\delta_{ij} = \delta_{\mu_i\mu_j}$ are introduced.

The normalization of the angular polynomials is defined by

$$\sum_{\{\mu\}} T^{(l)}_{\mu_1\ldots\mu_l}(n_1)T^{(l)}_{\mu_1\ldots\mu_l}(n_2) = \frac{1}{2^l}C^1_l(t), \qquad t = (n_1 n_2),$$

$$C^1_l(t) = \sqrt{\frac{\pi}{2}}p_l(t),$$

$$\int\limits_{-1}^{1} dt\ \sqrt{1-t^2}\cdot p_l(t)p_{l'}(t) = \int\limits_0^{\pi} d\gamma\ \sin^2\gamma\ p_l(\cos\gamma)p_{l'}(\cos\gamma) = \delta_{ll'}$$

where $C^1_l(t)$ are the Gegenbauer polynomials. In addition one can get

$$\int dn\ (T^{(l)}_{1\ldots l}(n_1)T^{(l)}_{1\ldots l}(n))\cdot(T^{(l)}_{1\ldots l}(n)T^{(l)}_{1\ldots l}(n_2))$$
$$= \frac{2\pi^2}{2^l(l+1)}(T^{(l)}_{1\ldots l}(n_1)T^{(l)}_{1\ldots l}(n_2)) = \frac{2\pi^2}{2^{2l}(l+1)}C^1_l((n_1 n_2))$$

B. Radial functions

The radial functions are defined in a standard way (see, for example, [19]). Let us define

$$c(n) = \frac{1}{2}\int\limits_0^{\infty} du\ u^{2+2n}K_1(u) = 4^n(n+1)!n!,$$

$$C_n(l) = \begin{vmatrix} c(l) & c(l+1) & \ldots & c(l+n) \\ c(l+1) & c(l+2) & \ldots & c(l+n+1) \\ \ldots & \ldots & \ldots & \ldots \\ c(l+n) & c(l+n+1) & \ldots & c(l+2n) \end{vmatrix}.$$

The radial polinomials read:

$$P_{(nl)}(u^2) = \frac{1}{\sqrt{D_{n-1}(l)D_n(l)}}$$
$$\cdot\begin{vmatrix} c(l) & c(l+1) & c(l+2) & \ldots & c(l+n) \\ c(l+1) & c(l+2) & c(l+3) & \ldots & c(l+n+1) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ c(l+n-1) & c(l+n) & c(l+n+1) & \ldots & c(l+2n-1) \\ 1 & u^2 & (u^2)^2 & \ldots & (u^2)^n \end{vmatrix}$$

where $D_n(l) = \det C_n(l)$.

The orthonormality condition is as follows:

$$\frac{1}{2}\int\limits_0^{\infty} du\ u^{2+2l}K_1(u)P_{(nl)}(u^2)P_{(n'l)}(u^2) = \delta_{nn'}.$$

The first four polynoms are

$$aP_{(00)} = a,$$
$$aP_{(10)}(ax) = \frac{a}{\sqrt{2}}\left[-1+\frac{a^2x^2}{8}\right],$$
$$aP_{(20)}(ax) = \frac{a}{\sqrt{26}}\left[3-\frac{5a^2x^2}{8}+\frac{(a^2x^2)^2}{96}\right],$$
$$aP_{(30)}(ax) = \frac{a}{\sqrt{4303}}\left[-34+\frac{37a^2x^2}{4}-\frac{53(a^2x^2)^2}{192}+\frac{13(a^2x^2)^3}{9216}\right].$$

C. Vertex functions

The vertex function

$$V_Q(k) = i^l \int dx \ \sqrt{D_m(x) D_a(x)} \ a P_Q(ax) e^{-ikx}$$

in the approximation

$$\sqrt{D_m(x) D_a(x)} \approx D_{\frac{m+a}{2}}(x)$$

looks as

$$V_Q(k) \approx i^l \int dx \ D_{\frac{m+\mu}{2}}(x) a P_Q(ax) e^{-ipx} = i^l P_Q\left(ia\frac{\partial}{\partial k}\right) \cdot \frac{a}{\left(\frac{m+a}{2}\right)^2 + k^2}.$$

The following equality is useful:

$$\left(\frac{\partial^2}{\partial k^2}\right)^s \cdot \frac{1}{M^2+k^2} = \Box_k^s \cdot \frac{1}{M^2+k^2}$$
$$= \sum_{j=0}^{s} (-1)^{s-j} \cdot \frac{4^s s!(s+1)!(s+j)!}{j!(j+1)!(s-j)!} \cdot \frac{(k^2)^j}{(M^2+k^2)^{s+j+1}}$$

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